Any one of the transformations $\Lambda^{-1}T_x\Lambda$ effects a translation on the three sides, AB, CD, DA of S_1 and carries the side BC into the curve to which T_2F_1 carries the linear segment in which the square is met by a line parallel to AD at a distance x from AD. Since if T_x is the translation which has the same effect as $\Lambda^{-1}T_x\Lambda$ on the point A, the transformation $T_x^{-1}\Lambda^{-1}T_x\Lambda$ leaves all points of the three edges AB, CD, DA of S_1 invariant. Denote $T_x^{-1}\Lambda^{-1}T_x\Lambda$ by F_x. The set of transformations F_x $(0 \leqq x \leqq 1)$ is obviously a continuous one-parameter family of $(1-1)$ continuous transformations; F_0 is the identity; and F_1 the given transformation already denoted by F_1. Hence F_1 is a deformation.

The last paragraph can be replaced by the observation that since the product of two deformations is a deformation, F_1, which is the product of T_2^{-1} and T_2F_1, must be a deformation. It seems worth while, however, to indicate, as has been done, something of the nature of the family of transformations F_x which the process sets up.

A THEOREM ON SERIES OF ORTHOGONAL FUNCTIONS WITH AN APPLICATION TO STURM-LIOUVILLE SERIES

By George D. Birkhoff

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated by E. H. Moore, October 10, 1917

1. *The Theorem.*—An infinite set of continuous functions $u_1(x), u_2(x), \ldots$ is *closed* on the interval $0 \leqq x \leqq 1$ if there exists no continuous function $f(x)$ not identically zero for which $\int_0^1 f(x)\, u_n(x)\, dx$ vanishes for all n; the set is *normalized* if $\int_0^1 u_n^2(x)\, dx = 1$ for all n; it is *orthogonal* if $\int_0^1 u_m(x)\, u_n(x) dx = 0$ for $m \neq n$. Most of the series of mathematical physics are linear in closed normalized orthogonal sets of functions.

THEOREM. *If $u_1(x), u_2(x), \ldots$ form a closed normalized orthogonal set of functions, and if $\bar{u}_1(x), \bar{u}_2(x), \ldots$ form a second normalized orthogonal set such that*

$$\sum_{n=1}^{\infty} (u_n(x) - \bar{u}_n(x))\, u_n(y) \qquad (0 \leqq x, y \leqq 1)$$

converges to a function $H(x, y)$ less than 1 in numerical magnitude in such wise that the series multiplied through by an arbitrary continuous function $f(x)$ can be integrated term by term as to x and yields a uniformly convergent series, then the set $\bar{u}_1(x), \bar{u}_2(x), \ldots$ is closed also.

Proof. If the set $\bar{u}_1(x), \bar{u}_2(x), \ldots$ is not closed there exists an f not identically zero such that $\int_0^1 f(x)\, \bar{u}_n(x)\, dx$ vanishes for all n. In this event, if we multiply through the equation of definition for H by $f(x)$ and integrate as we may by hypothesis, we find

$$\int_0^1 f(x)\, H(x, y)\, dx = \sum_{n=1}^{\infty} \int_0^1 f(x)\, u_n(x)\, dx \cdot u_n(y),$$

where the series on the right-hand side will converge uniformly by hypothesis and so represents a continuous function.

This series on the right has the value $f(y)$. In fact the difference

$$f(y) - \sum_{n=1}^{\infty} \int_0^1 f(x)\, u_n(x)\, dx \cdot u_n(y)$$

is a continuous function $\varphi(y)$ such that $\int_0^1 \varphi(y) u_n(y)\, dy$ vanishes for all n, precisely because the set $u_1(x), u_2(x), \ldots$ is normalized orthogonal. But the set $u_1(x), u_2(x), \ldots$ is closed by hypothesis; hence we infer that φ vanishes identically. Hence the right-hand member of the preceding equation has the value $f(y)$. That equation may now be written

$$\int_0^1 f(x)\, H(x, y)\, dx = f(y).$$

If the maximum numerical value of $f(x)$ for $0 \leqq x \leqq 1$ is $F \neq 0$, and if this value is taken on for $y = y_0$, we get

$$\left| \int_0^1 f(x)\, H(x, y_0)\, dx \right| = F.$$

But this is impossible since by hypothesis $|H| < 1$ and $|f| \leqq F$.

The set $\bar{u}_1(x), \bar{u}_2(x), \ldots$ is therefore closed also.

2. *An Application.* The Sturm-Liouville series, in a specialized but typical form, arises from the set of functions $u_1(x), u_2(x), \ldots$ which are the solutions of a linear differential equation of the second order

$$u'' + (\lambda - g(x))\, u = 0$$

satisfying boundary conditions $u(0) = u(1) = 0$, for the ordered set of parameter values $\lambda_1, \lambda_2, \ldots$ of λ respectively. We will assume that $g(x)$ and $dg(x)/dx$ are real and continuous.

By means of the methods of Sturm it is proved that $u_1(x), u_2(x), \ldots$ may be taken to form a normalized orthogonal set in which $u_n(x)$ will vanish precisely n times within the interval $0 \leqq x \leqq 1$. And the methods of Liouville give the asymptotic form of $u_n(x)$; it will be convenient for us to use the formula

$$u_n(x) = \sqrt{2}\left[\sin n\pi x + \frac{\varphi(x)\cos n\pi x}{n} + \frac{M_n(x, g(x))}{n^2}\right]$$

in which M_n is a continuous functional of x and $g(x)$, bounded as long as $g(x)$ and $dg(x)/dx$ are bounded uniformly for all n.

For the satisfactory investigation of the representation of an arbitrary function in Sturm-Liouville series it is necessary to know further that the set $u_1(x)$, $u_2(x)$, . . . thus obtained is closed. A very simple proof and the first is due to Stekloff.[1] His proof uses, however, the theory of functions of a complex variable. There is indeed a special case, namely the case when $g(x)$ vanishes identically and $u_n(x) = \sqrt{2} \sin n\pi x$, when a short proof is possible by an elementary method.

The theorem proved above makes possible an immediate demonstration that the set is closed, once the asymptotic formula above and the fact for the special case are granted. Let us replace $g(x)$ by $\sigma g(x)$ where σ is a real parameter varying from 0 to 1. If then $u_1(x)$, $u_2(x)$, . . . denote the members of the set for any particular σ and $\bar{u}_1(x)$, $\bar{u}_2(x)$, . . . denote the members for another value of σ, say $\bar{\sigma}$, it will suffice to show that the series for H will have the properties demanded in the theorem for $|\sigma - \bar{\sigma}| \leqq \delta > 0$. For in this event since the set is closed for $\sigma = 0$ (the special case), it will be closed for $\sigma \leqq \delta$ by an application of the theorem; by successive further applications of the theorem it is shown in the same way that the set is closed for $\sigma \leqq 2\delta$, $\sigma \leqq 3\delta$, . . . , so that finally we infer that the theorem is true for $\sigma = 1$, i.e. for the given set.

In virtue of the explicit formula for $u_n(x)$ we have

$$u_n(x) - \bar{u}_n(x) = \sqrt{2}\left[\frac{(\sigma - \bar{\sigma})\,\varphi(x)\cos n\pi x}{n} + \frac{M_n(x, \sigma g(x)) - M_n(x, \bar{\sigma} g(x))}{n^2}\right],$$

$$u_n(y) = \sqrt{2}\left[\sin n\pi y + \frac{\varphi(y)\cos n\pi y}{n} + \frac{M(y, \sigma g(x))}{n^2}\right].$$

Multiplying these two expressions on the right together we obtain the typical term of the H series. This series breaks up at once into six (2×3) other series all of which converge uniformly to a small value for $|\sigma - \bar{\sigma}|$ small save the series obtained from the leading terms in both expressions, namely

$$2(\sigma - \bar{\sigma})\,\varphi(x)\sum_{n=1}^{\infty}\frac{\cos n\pi x \sin n\pi y}{n}.$$

If we omit the small factor $(\sigma - \bar{\sigma})\,\varphi(x)$, this series may be written

$$\sum_{n=1}^{\infty}\frac{\sin n\pi(x+y) - \sin n\pi(x-y)}{n}.$$

Now the sum of any number of terms of a series Σ (sin $n\pi z$)/n is known to remain uniformly bounded,[2] and the series is known to converge everywhere, uniformly save in the immediate vicinity of the values $z = 0$, $\pm 2\pi$, $\pm 4\pi$, etc. Hence the series displayed above converges uniformly for all x, y save in the immediate vicinity of $x = y$, where, however, the sum of any number of terms of the series is bounded. Hence all six types of series will converge to values small numerically for $|\sigma - \bar{\sigma}|$ small, and, when multiplied through by any continuous function may be integrated term by term as to x, yielding uniformly convergent series in y. Thus the H series will have the stated properties for $|\sigma - \bar{\sigma}| \leqq \delta > 0$.

3. *Generalization.*—The theorem suggests at once a theorem in General Analysis as defined by E. H. Moore.[3] If we employ a quasi-geometrical terminology this generalization may be stated as follows: *any set of orthogonal vectors in a functional space lying near enough to a complete set of orthogonal vectors in that space is itself complete.* Another still wider generalization suggests itself: *any set of vectors in a functional space lying near enough to a complete set of vectors admitting a reciprocal set is itself complete and admits a reciprocal set.*[4] This second generalization evidently plays the same part in relation to biorthogonal sets that the first does for orthogonal sets.

[1] See Kneser, *Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik*, Braunschweig, 1911, pp. 84–95.

[2] See D. Jackson, *Rend. Circ. Mat. Palermo*, **32**, 1911, (257–262).

[3] See *Bull. Amer. Math. Soc., New York*, **18**, 1911–1912, (334–362).

[4] For a theorem of this type see an article of mine, *Paris, C.-R. Acad. Sci.*, **161**, 1917, (942–945).

LOW-TEMPERATURE FORMATION OF ALKALINE FELDSPARS IN LIMESTONE

By Reginald A. Daly

DEPARTMENT OF GEOLOGY AND GEOGRAPHY, HARVARD UNIVERSITY

Communicated by W. M. Davis, October 5, 1917

Recent monographs by A. Heim[1] and D. Trümpy,[2] dealing with certain rock formations in Switzerland, put new emphasis on an important problem in minerogenesis. At different horizons of the Jurassic limestones of the Churfirsten-Mattstock mountain group, Heim has found abundant crystals of albite which have evidently developed *in situ* and are not of clastic origin. The crystals are automorphic, with maximum lengths of 0.2 mm. and average lengths much less (see fig. 150 in Heim's memoir). Minute crystals of ankerite are associated. Both albite and ankerite are regarded by Heim as due to crystallization on the sea-floor, during